

ZASSENHAUS CONJECTURE FOR CYCLIC-BY-ABELIAN GROUPS

MAURICIO CAICEDO, LEO MARGOLIS, AND ÁNGEL DEL RÍO

ABSTRACT. Zassenhaus Conjecture for torsion units states that every augmentation one torsion unit of the integral group ring of a finite group G is conjugate to an element of G in the units of rational group algebra $\mathbb{Q}G$. This conjecture has been proved for nilpotent groups, metacyclic groups and some other families of groups. We prove the conjecture for cyclic-by-abelian groups.

In this paper G is a finite group and RG denotes the group ring of G with coefficients in a ring R . The units of RG of augmentation one are usually called normalized units. In the 1960s Hans Zassenhaus established a series of conjectures about the finite subgroups of normalized units of $\mathbb{Z}G$. Namely he conjectured that every finite group of normalized units of $\mathbb{Z}G$ is conjugate to a subgroup of G in the units of $\mathbb{Q}G$. These conjecture is usually denoted (ZC3), while the version of (ZC3) for the particular case of subgroups of normalized units with the same cardinality as G is usually denoted (ZC2). These conjectures have important consequences. For example, a positive solution of (ZC2) implies a positive solution for the Isomorphism and Automorphism Problems (see [Seh93] for details). The most celebrated positive result for Zassenhaus Conjectures is due to Weiss [Wei91] who proved (ZC3) for nilpotent groups. However Roggenkamp and Scott founded a counterexample to the Automorphism Problem, and henceforth to (ZC2) (see [Rog91] and [Kli91]). Later Hertweck [Her01] provided a counterexample to the Isomorphism Problem.

The only conjecture of Zassenhaus that is still up is the version for cyclic subgroups namely:

Zassenhaus Conjecture for Torsion Units (ZC1). If G is a finite group then every normalized torsion unit of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G .

Besides the family of nilpotent groups, (ZC1) has been proved for some concrete groups [BH08, BHK04, HK06, LP89, LT91, Her08b], for groups having a Sylow subgroup with an abelian complement [Her06], for some families of cyclic-by-abelian groups [LB83, LT90, LS98, MRSW87, PMS84, PMRS86, dRS06, RS83] and some classes of metabelian groups not necessarily cyclic-by-abelian [MRSW87, SW86]. Other results on Zassenhaus Conjectures can be found in [Seh93, Seh01] and [Seh03, Section 8].

The latest and most general result for (ZC1) on the class of cyclic-by-abelian groups is due to Hertweck [Her08a] who proved (ZC1) for finite groups of the form $G = AX$ with A a cyclic normal subgroup of G and X an abelian subgroup of

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G . This includes the class of metacyclic groups that was not covered in previous results. The aim of this paper is to prove (ZC1) for arbitrary cyclic-by-abelian groups. Formally we prove

Theorem. Let G be a finite cyclic-by-abelian group. Then every normalized torsion unit of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G .

Our strategy uses induction on the order of the group G and on the order of the torsion unit. In other words we consider a finite cyclic-by-abelian group G , which is a minimal counterexample to (ZC1), and u a torsion unit in $\mathbb{Z}G$, which is a minimal counterexample to (ZC1). Here minimal means “of minimal order”. In particular, we assume that (ZC1) holds for proper subgroups and quotients of G and for units in proper subgroups of the group generated by u .

Most of the ideas used in this paper are either due to or inspired from the techniques introduced by Hertweck in [Her08a] which we have adapted in some steps of the proof to avoid some difficulties appearing in the general case which are not encountered in the hypothesis of [Her08a]. For example, the strategy in [Her08a] for the case when $G = AX$ with A cyclic normal and X abelian, consists in first proving (ZC1) for torsion units u with augmentation 1 modulo A , with the help of a result of Cliff and Weiss for the matrix version of Zassenhaus Conjecture (Theorem 1.5) and a beautiful use of Weiss permutation module theorem [Wei88], then using this to prove the results for units with augmentation 1 modulo $C_G(A)$ and then reducing the general case to this special case. In several steps of the proof one uses a faithful linear representation of A and the fact that $C_G(A) = AC_X(A) = AZ(G)$ (and hence $C_G(A)$ is cyclic-by-central). In the words of Hertweck the last fact is “the main reason for assuming that A is covered by an abelian subgroup—rather than assuming that G/A is abelian”.

In our strategy the subgroup $D = Z(C_G(A))$, for A a cyclic subgroup of G containing G' , plays a very important role. We first prove (ZC1) for units with augmentation 1 modulo D using local methods over the p -adic integers and then we prove (ZC1) for the remaining units using the so called Luthar-Passi Method. As D is not cyclic-by-central it is not possible to use neither Cliff-Weiss Theorem, nor a faithful linear character of an appropriate cyclic subgroup of G . Instead, for the first part of the proof we adapt the p -adic methods of Hertweck and Cliff-Weiss to our situation with a careful revision of their proofs, using linear characters of D with kernels not intersecting A , and in the second part we use a family of linear characters of D with kernel not containing any normal subgroup of G . This introduces some difficulties in the arguments which makes the proofs more involved than in [Her08a].

1. NOTATION, PRELIMINARIES AND SOME TOOLS

In this section we establish the general notation and collect some known results, which will be used throughout the paper.

As it is customary φ denotes Euler’s totient function. The cardinality of a set X is denoted $|X|$. For every integer n we let ζ_n denote a fixed complex primitive root of unity of order n . The ring of p -adic integers, for p a prime integer, is denoted \mathbb{Z}_p .

We use the standard group theoretical notation. In particular, if G is a group, then $Z(G)$ denotes the center of G , G' the commutator subgroup of G and $\exp(G)$

the exponent of G . If $g, h \in G$ then $|g|$ denotes the order of g , $g^h = h^{-1}gh$, $(g, h) = g^{-1}g^h = g^{-1}h^{-1}gh$ and g^G denotes the conjugacy class of g in G . If $X \subseteq G$ then $\langle X \rangle$ denotes the subgroup generated by X , $C_G(X) = \{g \in G : (x, g) = 1 \text{ for every } x \in X\}$, the centralizer of X in G and $N_G(X) = \{g \in G : X^g \subseteq X\}$, the normalizer of X in G . Let p be a prime integer. If g has finite order then g_p and $g_{p'}$ denote the p -part and p' -part of g , respectively. If G has a unique p -Sylow subgroup (respectively a unique p' -Hall subgroup) then it is denoted G_p (respectively $G_{p'}$).

In the remainder of this section R stands for a commutative ring. If N is a normal subgroup of G then the N -augmentation map of RG is the unique ring homomorphism $\omega_N : RG \rightarrow R(G/N)$ extending the natural map $G \rightarrow G/N$ and acting on R as the identity. In particular $\omega = \omega_G$ is the augmentation map of RG . Let $r = \sum_{g \in G} r_g g \in RG$ with $r_g \in R$ for every g . For every $g \in G$, let $\varepsilon_g^G(r)$ denote the partial augmentation of r in the conjugacy class of g in G , that is

$$\varepsilon_g^G(r) = \sum_{h \in g^G} r_h.$$

If the group G is clear from the context we simply write $\varepsilon_g(r)$. Conjugacy classes in RG and partial augmentation are strongly related. We collect in the following remark some easy facts about this relation.

Remark 1.1. If $x \in G$ then $\varepsilon_x^G : RG \rightarrow R$ is an R -linear map which satisfies $\varepsilon_x^G(uv) = \varepsilon_x^G(vu)$ for every $u, v \in RG$. Using this it is easy to prove that if $u \in RG$ and $g \in G$ are conjugate in RG , then $\varepsilon_g^G(u) = 1$ and $\varepsilon_x^G(u) = 0$ for every $x \in G$ with $x \notin g^G$. Hence, for such u and g and a normal subgroup N of G we have $\omega_N(u) \neq 0$ if and only if $g \in N$ and in that case $\omega_N(u) = 1$.

One of the main tools to study (ZC1) is the following well known result which is somehow a converse of Remark 1.1 (see e.g. [Seh93, Lemma 41.5]).

Proposition 1.2. *Let u be a normalized torsion unit of $\mathbb{Z}G$. Then u is conjugate in $\mathbb{Q}G$ to an element of G if and only if $\varepsilon_g^G(v) \geq 0$ for every $v \in \langle u \rangle$ and every $g \in G$.*

Proposition 1.2 is commonly presented in the following equivalent form: A normalized torsion unit u of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G if and only if for every $v \in \langle u \rangle$, there is $g \in G$ such that for every $x \in G$ we have $\varepsilon_x^G(v) \neq 0$ if and only if $x \in g^G$.

The following proposition collects some results from [Her06] and [Her08a] which will be very useful in our arguments.

Proposition 1.3. *Let G be a finite group and p a prime integer.*

- (1) *Let R be a p -adic ring with quotient field K and u a normalized torsion unit of RG .*
 - (a) *Suppose $\omega_P(u) = 1$ for P a normal p -subgroup of G . Then u is conjugate in KG to an element of P .*
 - (b) *Suppose that the p -part of u is conjugate to an element x of G in the units of RG and g is an element of G such that the p -parts of x and g are not conjugate in G . Then $\varepsilon_g(u) = 0$.*
- (2) *Let u be a torsion unit of $\mathbb{Z}G$.*
 - (a) *If $\varepsilon_g(u) \neq 0$ with $g \in G$ then the order of g divides the order of u .*

- (b) Assume that $\omega_P(u) = 1$ with P a cyclic normal p -subgroup of G . Then u is conjugate in $\mathbb{Q}G$ to an element $x \in P$. If moreover, $C_G(x)$ has a normal p -complement, then u and x are conjugate in $\mathbb{Z}_p G$.

We now present a matrix version of (ZC1) which was introduced in [MRSW87] as a strategy to prove (ZC1) in some cases. If k is a positive integer then the action of ω on the matrix entries defines a ring homomorphism $M_k(RG) \rightarrow M_k(R)$. It restricts to a group homomorphism $\mathrm{GL}_k(RG) \rightarrow \mathrm{GL}_k(R)$. Following [CW00] we let $\mathrm{SGL}_k(RG)$ denote the kernel of this group homomorphism. The matrix version of (ZC1) is the following problem:

Problem 1.4. *Let G be a finite group. Is every element of finite order of $\mathrm{SGL}_k(\mathbb{Z}G)$ conjugate in $\mathrm{GL}_k(\mathbb{Q}G)$ to a diagonal matrix with diagonal entries in G ?*

Cliff and Weiss solved this problem for nilpotent groups and arbitrary k .

Theorem 1.5. [CW00] *If G is a finite nilpotent group then Problem 1.4 has a positive solution for every $k \geq 1$ if and only if G has at most one non-cyclic Sylow subgroup.*

Let N be a normal subgroup of G with $k = [G : N]$. Then RG is a (RG, RN) -bimodule and the right RN -module RG is free with a basis formed by a transversal of N in G . For every $r \in RG$ let $\rho_N(r)$ denote the matrix representation of left multiplication by r with respect to this basis. This defines a ring homomorphism $\rho_N : RG \rightarrow M_k(RN)$. If r is a unit of RG such that $\omega_N(r) = 1$ then $\rho_N(r) \in \mathrm{SGL}_k(RN)$. If $x \in N$ then, by [Seh93, Lemma 41.10], we have

$$(1.1) \quad \varepsilon_x^N(\mathrm{tr}(\rho_N(r))) = [C_G(x) : C_N(x)] \varepsilon_x^G(r),$$

where $\mathrm{tr}(U)$ stands for the trace of a matrix $U \in M_k(RN)$. If N is not commutative and $U, V \in M_k(RN)$ then $\mathrm{tr}(UV)$ and $\mathrm{tr}(VU)$ might be different. However, $\mathrm{tr}(UV) - \mathrm{tr}(VU) \in [RN, RN]$, where $[RN, RN]$ is the R linear span of the Lie products $[a, b] = ab - ba$, with $a, b \in N$. Hence $\varepsilon_x^N(\mathrm{tr}(UV)) = \varepsilon_x^N(\mathrm{tr}(VU))$ for every $x \in N$. In particular, if U and V are conjugate in $M_k(RN)$, then $\varepsilon_x^N(\mathrm{tr}(U)) = \varepsilon_x^N(\mathrm{tr}(V))$.

A useful technique to deal with Zassenhaus conjecture (in the most general form) is the so called double action formalism introduced by Weiss [Wei88]. Let G and H be groups, R a commutative ring and let $\alpha : H \rightarrow \mathrm{GL}_k(RG)$ be a group homomorphism. Then M^α denotes the right $R(H \times G)$ module $(RG)^k$ with multiplication defined by $x \cdot r(h, g) = \alpha(h)^{-1} x r g$ for $x \in (RG)^k$, $g \in G$, $h \in H$ and $r \in R$. It is easy to see that if $\beta : H \rightarrow \mathrm{GL}_k(RG)$ is another group homomorphism then $M^\alpha \cong M^\beta$ if and only if α and β are conjugate in $\mathrm{GL}_k(RG)$, i.e. if and only if there exist $u \in \mathrm{GL}_k(RG)$ such that $\beta(h) = \alpha(h)^u$ for every $h \in H$. For example, if $C_m = \langle c \rangle$, the cyclic group of order m generated by c , and u and v are torsion unit of order m in $\mathrm{GL}_k(RG)$ then u and v are conjugate in $\mathrm{GL}_k(RG)$ if and only if the $R(C_m \times G)$ modules M^{α_u} and M^{α_v} are isomorphic, where α_u and α_v are the homomorphisms $C_m \rightarrow \mathrm{GL}_k(RG)$ mapping c to u and v respectively.

Let m be a positive integer, set $\Gamma = C_m \times G$ and let $G[m]$ denote a set of representatives of G -conjugacy classes of elements $g \in G$ with $g^m = 1$. For every $g \in G$ let $[g] = \langle \langle c, g \rangle \rangle \leq \Gamma$. For every prime integer p and $g \in G$ let $G_{g,p}[m] = \{h \in G[m] : g_p \text{ and } h_p \text{ are conjugate in } G\}$. For H a subgroup of a group K we let $\mathrm{ind}_H^K 1$ denote the character of K induced from the trivial character of H .

Lemma 1.6. *Let G be a finite nilpotent group, $u \in \text{SGL}_k(\mathbb{Z}G)$ with $u^m = 1$ and let χ denote the character of M^{α_u} . Then*

- (1) *For every $g \in G[m]$ we have $\chi(c, g) \in |C_G(g)|\mathbb{Z}$ and $\chi = \sum_{g \in G[m]} \frac{\chi(c, g)}{|C_G(g)|} \text{ind}_{[g]}^\Gamma 1$.*
- (2) *For every prime p and every $g \in G[m]$, $\sum_{h \in G_{g,p}[m]} \frac{\chi(c, h)}{|C_G(h)|} \text{ind}_{[h_{p'}]}^{\Gamma_{p'}} 1$ is a proper character of $\Gamma_{p'}$.*

Proof. Cliff and Weiss [CW00] prove that $\chi = \sum_{g \in G[m]} a_g \text{ind}_{[g]}^\Gamma 1$ for unique integers a_g and $\sum_{h \in G_{g,p}[m]} a_h \text{ind}_{[h_{p'}]}^{\Gamma_{p'}} 1$ is a proper character of $\Gamma_{p'}$. So we only have to prove that $\chi(c, g) = a_g |C_G(g)|$ and this follows from

$$(\text{ind}_{[h]}^\Gamma 1)(c, g) = \begin{cases} |C_G(g)|, & \text{if } h \in g^G; \\ 0, & \text{otherwise.} \end{cases}$$

□

2. TORSION UNITS WITH D -AUGMENTATION 1

In this section G is a finite group. The title of this section refers to $D = Z(C_G(A))$ for a cyclic subgroup A of G containing G' (see the introduction). So the aim of this section is to prove (ZC1) for torsion units u with $\omega_D(u) = 1$.

We start with a lemma, which seems to be folklore.

Lemma 2.1. *Let N be a nilpotent normal subgroup of G and u a torsion unit of $\mathbb{Z}G$ such that $\omega_N(u) = 1$. Then*

- (1) *every prime divisor of the order of u divides the order of N and*
- (2) *if the order of u is a power of a prime p , then $\omega_{N_p}(u) = 1$.*

Proof. (1) We argue by induction on the number of primes dividing $|N|$. If this number is 0 then $N = 1$ and hence $u = \omega_N(u) = 1$. Assume that $|N|$ is divisible by p . By hypothesis $\omega_{N/N_p}(\omega_{N_p}(u)) = \omega_N(u) = 1$. Let n be the order of $\omega_{N_p}(u)$. By the induction hypothesis every prime divisor of n divides $[N : N_p]$. Moreover the order of u^n is a power of p by [Seh93, Lemma 7.5]. Let q be a prime divisor of $|u|$. Then q is either p or a divisor of n . We conclude that q divides $|N|$.

(2) Assume that the order of u is a power of p and set $u_1 = \omega_{N_p}(u)$. Then u_1 is a p -element of $\mathbb{Z}(G/N_p)$ such that $\omega_{N/N_p}(u_1) = \omega_N(u) = 1$. Since p is coprime with $[N : N_p]$, the order of u_1 is coprime with p , by (1). Hence $u_1 = 1$, as desired. □

The following lemma extends [Her08a, Claim 5.2] where it was proved for the case where $G = AX$ with A a cyclic normal subgroup of G and X an abelian subgroup of G .

Lemma 2.2. *Let A be a cyclic subgroup of G containing G' and let N be a non-trivial p -subgroup of A for some prime p . Then $C_G(N)$ has a normal p -complement (i.e. it has a normal p' -Hall subgroup).*

Proof. By replacing G by $C_G(N)$ we may assume without loss of generality that N is central in G . If N_1 is the unique minimal non-trivial subgroup of N and $\alpha : \text{Aut}(A_p) \rightarrow \text{Aut}(N_1)$ is the restriction map, then the kernel of α is a p -group. Therefore the p' -elements of G commute with the p -elements of A . Let H be a p' -Hall subgroup of G . We will prove that H is normal in G . Let $h \in H$ and $g \in G$. As G/A is abelian, $h^g = ah$ for some $a \in A$. By Hall Theorem [Rob82, 9.1.7], it easily follows, that $A_{p'} \subseteq H$. Therefore $a_{p'}h \in H$ and in particular the order of

$a_{p'}h$ is coprime with p . Thus $(a_p, a_{p'}h) = 1$ and hence the order of h^g is divisible by the order of a_p . Thus $a_p = 1$ and we conclude that $h^g = a_{p'}h \in H$. \square

The argument of the following lemma was already used in [dRS06] and [Her08a] to prove that partial augmentations of elements in $G \setminus C_G(A)$ are non-negative in the minimal counterexamples. We need this also for elements in $G \setminus Z(C_G(A))$.

Lemma 2.3. *Let A be a cyclic subgroup of G containing G' and assume that G/N satisfies (ZC1) for every non-trivial subgroup N of A . Let u be a normalized torsion unit in $\mathbb{Z}G$ and let $x \in G \setminus (Z(C_G(A)))$. Then $\varepsilon_x(u) \geq 0$.*

Proof. Let $C = C_G(A)$ and $D = Z(C)$. If $x \notin C$, then $N = \langle \langle A, x \rangle \rangle$ is a non-trivial subgroup of G contained in A . By hypothesis, (ZC1) holds for G/N . Hence $\varepsilon_{xN}^{G/N}(\omega_N(u)) \geq 0$, by Proposition 1.2. By [dRS06, Lemma 2], $Nx^G = x^G$ and this implies that $\varepsilon_x^G(u) = \varepsilon_{xN}^{G/N}(\omega_N(u)) \geq 0$, as desired.

Assume that $x \in C \setminus D$. Then $(x, c) \neq 1$ for some $c \in C$. Let $N = \langle \langle x, c \rangle \rangle$, a non-trivial normal subgroup of G . We claim that $x^G = Nx^G$. Indeed, as $G' \subseteq A$ and $(A, C) = 1$, if $w, v \in C$ and $g \in G$, then $(wv, g) = (w, g)^v(v, g) = (w, g)(v, g)$. Thus, if $n \in N$, then $n = (x, c)^m = (x, c^m)$ for some integer m . If $g \in G$, then $nx^g = (x, c^m)x^g = (x, c^m)x(x, g) = x(x, gc^m) = x^{gc^m}$. This proves the claim.

We set $\bar{\alpha} = \omega_N(\alpha)$ for every $\alpha \in \mathbb{Z}G$. By hypothesis (ZC1) holds for G/N and hence $\varepsilon_{\bar{x}}^{G/N}(\bar{u}) \geq 0$. If $u = \sum_{g \in G} u_g g$ with $u_g \in \mathbb{Z}$ for each g then $\varepsilon_x^G(u) = \sum_{g \in x^G} u_g = \sum_{g \in Nx^G} u_g = \varepsilon_{\bar{x}}^{G/N}(\bar{u}) \geq 0$, as desired. \square

The following lemma comes from a closer investigation of the proof of Theorem 1.5 as given [CW00].

Lemma 2.4. *Let N be an abelian normal subgroup of G and u a torsion unit in $\mathbb{Z}G$ with $\omega_N(u) = 1$. Let η be an irreducible character of N and $n \in N$. Then*

$$\sum_{h \in \ker \eta} [C_G(hn) : N] \varepsilon_{hn}^G(u) \geq 0.$$

Proof. Let m be the order of u and let v be the image of u under the natural permutation homomorphism $\mathbb{Z}G \rightarrow GL_k(\mathbb{Z}N)$. Then $v \in SGL_k(\mathbb{Z}N)$, where $k = [G : N]$. Let $\Gamma = C_m \times N$, with $C_m = \langle c \rangle$, a cyclic group of order m , let χ be the character of M^{α_v} and let $N[m] = \{n \in N : n^m = 1\}$. By [Wei91, Lemma 1] we know $|C_G(n)|\varepsilon_n^G(u) = \chi(c, n)$ for every $n \in N$. Moreover, $\varepsilon_h^G(u) = 0$ if $h \in N \setminus N[m]$, by statement (2a) of Proposition 1.3. Combining this with Lemma 1.6 we have

$$\begin{aligned} \chi &= \sum_{h \in N[m]} \frac{\chi(c, h)}{|C_N(h)|} \text{ind}_{[h]}^\Gamma 1 = \sum_{h \in N[m]} [C_G(h) : N] \varepsilon_h^G(u) \text{ind}_{[h]}^\Gamma 1 \\ &= \sum_{h \in N} [C_G(h) : N] \varepsilon_h^G(u) \text{ind}_{[h]}^\Gamma 1 \end{aligned}$$

and for every prime integer p and every character ψ of $\Gamma_{p'}$ we have

$$\begin{aligned}
0 &\leq \left\langle \sum_{h \in N_{n,p}[m]} [C_G(h) : N] \varepsilon_h^G(u) \text{ind}_{[h_{p'}]}^{\Gamma_{p'}} 1, \psi \right\rangle_{\Gamma_{p'}} \\
&= \sum_{h \in N_{n,p}[m]} [C_G(h) : N] \varepsilon_h^G(u) \langle \text{ind}_{[h_{p'}]}^{\Gamma_{p'}} 1, \psi \rangle_{\Gamma_{p'}} \\
&= \sum_{h \in N, h_p = n_p, \psi(c_{p'}, h_{p'}) = 1} [C_G(h) : N] \varepsilon_h^G(u) = \sum_{h \in N_{p'}, \psi(1, h) = 1} [C_G(hh') : N] \varepsilon_{hh'}^G(u),
\end{aligned}$$

where h' is a fixed element of N with $h'_p = n_p$ and $\psi(c, h') = 1$.

Let ψ be the character of Γ given by $\psi|_N = \eta$ and $\psi(u) = \eta(n)^{-1}$. Then $\psi(c, n) = 1$ and therefore applying the previous inequality for $h' = h_p n$ with $h_p \in N_p$ we deduce that

$$\sum_{h \in N_{p'} \cap \ker \eta} [C_G(hh_p n) : N] \varepsilon_{hh_p n}^G(u) \geq 0.$$

We conclude that

$$\sum_{h \in \ker \eta} [C_G(hn) : N] \varepsilon_{hn}^G(u) = \sum_{h_p \in N_p \cap \ker \eta} \sum_{h_{p'} \in N_{p'} \cap \ker \eta} [C_G(h_p h_{p'} n) : N] \varepsilon_{h_p h_{p'} n}^G(u) \geq 0,$$

as desired. \square

The following theorem is an adjustment of [Her08a, Theorem 5.1] to our situation.

Theorem 2.5. *Let G be a finite group and A a cyclic normal subgroup of G containing G' . Set $D = Z(C_G(A))$ and let u be a torsion unit of $\mathbb{Z}G$ with $\omega_D(u) = 1$. If the order of u is a power of a prime p , then u is conjugate in $\mathbb{Z}_p G$ to an element of D_p .*

Proof. By statement (1a) of Proposition 1.3, u is conjugate in $\mathbb{Q}G$ to an $x \in D_p$. Let R be a p -adic ring with quotient field K containing a root of unity of order the exponent of G . We will prove that u is conjugate to x in RG . Then by [CR62, 30.25] the conjugation already takes place in $\mathbb{Z}_p G$.

Set $L = C_G(D_p)$, $E = C_G(x)$ and let Q be the normal p -complement of L , that exists by Lemma 2.2. Note that L and E are normal in G , for they contain A , and Q is also normal in G for it is a characteristic subgroup of L .

The primitive central idempotents of KQ belong to RQ , because the order of Q is invertible in R . Moreover G acts on these primitive central idempotents by conjugation. Let $\epsilon_1, \dots, \epsilon_\beta$ be the sums of the G -orbits of this action. Then $RG = \prod_{i=1}^{\beta} \epsilon_i RG$ and therefore it is enough to show that $\epsilon_i u$ is conjugate to $\epsilon_i x$ in $\epsilon_i RG$ for every i . Note that $\epsilon_i u$ is conjugate to $\epsilon_i x$ in $\epsilon_i KG$ and a primitive idempotent of KQ stays primitive in KL by Greens Indecomposability Theorem, since L/Q is a p -group [CR62, 19.23].

So fix one primitive central idempotent f of KQ and let ϵ be the sum of the G -conjugates of f . We have to prove that ϵu and ϵx are conjugate in ϵRG . Let e be the sum of different L -conjugates of f and write $e = e_1 + \dots + e_m$ with

orthogonal primitive idempotents of RQ . Let $T = C_G(e)$ and let $\{1 = s_1, \dots, s_n\}$ be a transversal of G/T . Then

$$\epsilon = \sum_{i=1}^m \sum_{j=1}^n e_i^{s_j^{-1}}$$

is a decomposition into primitive orthogonal idempotents of ϵRL . So by [Her08a, Lemma 4.6] there exist $g_{ij} \in G$ such that

$$v = \sum_{i=1}^m \sum_{j=1}^n e_i^{s_j^{-1}} x^{g_{ij}}$$

is conjugate to ϵu in RG . In particular $|v| = |\epsilon u| = |\epsilon x|$. Let $C = \langle c \rangle$ a cyclic group with the same order as v and consider the $R(C \times G)$ -modules $M = M^\alpha$ and $N = M^\beta$, with $\alpha = \alpha_v$ and $\beta = \alpha_{\epsilon x}$. (We are abusing the notation by writing α_v and $\alpha_{\epsilon x}$ instead of $\alpha_{v+(1-\epsilon)}$ and $\alpha_{\epsilon x+(1-\epsilon)}$, respectively.) We have to show that v is also conjugate to ϵx in ϵRG or equivalently that M and N are isomorphic as $R(C \times G)$ -modules.

As $x \in D_p$ and $e_i \in RL$ and both D_p and L are normal in G , every G -conjugate of x belongs to D_p and every G -conjugate of e_i belong to RL . On the other hand $(D_p, L) = 1$ and therefore every G -conjugate of x commutes with every G -conjugate of e_i . Using this it easily follows that each $RGe_i^{s_j^{-1}}$ is a submodule of both M and N . We set $M_{ij} = Me_i^{s_j^{-1}}$ and $N_{ij} = Ne_i^{s_j^{-1}}$, i.e. both M_{ij} and N_{ij} are $RGe_i^{s_j^{-1}}$, considered as submodules of M and N respectively. The strategy of the proof consists in pairing isomorphic M_{ij} 's and N_{ij} 's.

Firstly observe that if $g \in G$, then every two primitive idempotents ε_1 and ε_2 of RQe^g are conjugate in RLe^g , since KLe^g is simple and hence $RL\varepsilon_1 \cong RL\varepsilon_2$ (see e.g. Theorem 6.7, Proposition 16.16 and Problem 6.14 in [CR62]). If $e_i^{s_j^{-1}} = e_1^w$ with w a unit in RQe then $a \mapsto aw$ is an isomorphism $N_{1j} \rightarrow N_{ij}$. Secondly, if $q \in E$, then the map $a \mapsto aq$ is an isomorphism $Ne_i \rightarrow Ne_i^q$. Therefore, if we choose the transversal $\{s_j : 1 \leq j \leq n\} = \{h_{j_2}q_{j_1} \mid 1 \leq j_1 \leq n_1, 1 \leq j_2 \leq n_2\}$, with $\{h_1, \dots, h_{n_2}\}$ a transversal of G/TE and $\{q_1, \dots, q_{n_1}\}$ a transversal of $E/E \cap T$ (which is also a transversal of TE/T), and denote $N_{ij} = N_{i(j_1, j_2)}$ if $s_j = h_{j_2}q_{j_1}$. Then we have

$$(2.2) \quad N = \bigoplus_{i,j} N_{ij} \cong \bigoplus_{j_2=1}^{n_2} (N_{1(1, j_2)})^{mn_1}$$

If we have $g_{ij}s_j \equiv s_{j_0} \pmod{T}$ then $N_{ij_0} \cong M_{ij}$ via $a \rightarrow ag_{ij}$. So if $g_{ij}s_j \equiv h_{j_2} \pmod{TE}$ we can pick a suitable q_{j_1} such that by setting $s_{j_0} = h_{j_2}q_{j_1}$ we have $g_{ij}s_j \equiv s_{j_0} \pmod{T}$ and this gives $M_{ij} \cong N_{1(1, j_2)}$.

Set $X_{j_2} = \{(i, j) \mid g_{ij}s_j \equiv h_{j_2} \pmod{TE}\}$. By the previous paragraph, the isomorphism $M \cong N$ will follow from (2.2) provided $|X_{j_2}| = mn_1$. The remainder of the proof is dedicated to prove this equality. For this we will use a representation of ϵKG and investigate the multiplicities of eigenvalues of ϵx and v under this representation. They are the same for ϵx and v are conjugate in ϵKG . This is also the strategy in the proof of [Her08a, Theorem 5.1]. However the representation used by Hertweck was constructed using a faithful linear representation of A_p and here we need to use a linear representation of D_p . This representation cannot be faithful if D_p is not cyclic. Instead we use a subgroup H of G such that G/H is

cyclic and $H \cap A = 1$ and then consider a linear representation of D_p with kernel H . The existence of such H follows easily by observing that if H is a maximal subgroup of D_p not intersecting A , then D_p/H is cyclic because otherwise D_p/H contains a direct product $\langle c \rangle \times \langle d \rangle$ of cyclic groups of order p . This would yield to a contradiction using the maximality of H and the fact that A is cyclic.

Let π be the linear character of a representation of D_p with kernel H and let ψ be the sum of all irreducible characters of Q , that do not vanish on e . So $\psi(f) = 1$ for every primitive idempotent f of RQ satisfying $ef \neq 0$ and hence $\psi(1) = m$. Let ρ be a representation of $Q \times D_p$ affording $\psi \otimes \pi$. This can be chosen satisfying $\rho(e_i) = E_i$, where E_i denotes the elementary matrix with 1 in the i -th diagonal entry and 0 anywhere else. Then $\rho(e_i y) = \pi(y)E_i$ for any i and $y \in D_p$. Let $\chi = \text{ind}_{Q \times D_p}^T(\psi \otimes \pi)$, let Δ be a representation of T affording χ and let $\{t_1, \dots, t_k\}$ be a transversal of $T/Q \times D_p$. Then, after a suitable conjugation one may assume that

$$\Delta \left(\sum_{i=1}^m e_i y_i \right) = \text{diag} \left(\pi \left(y_i^{t_j} \right) \mid 1 \leq i \leq m, 1 \leq j \leq k \right) \in M_{mk}(K)$$

for every $y_1, \dots, y_m \in D_p$. Denote by $\bar{\Delta} : M_n(eKT) \rightarrow M_{nmk}(K)$ the map which acts like Δ componentwise.

As right KG -modules we have $eKG = \sum_{j=1}^n e^{s_j^{-1}} KG \cong (eKG)^n$ and so

$$(2.3) \quad eKG \cong \text{End}_{KG}(eKG) \cong M_n(\text{End}_{KG}(eKG)) \cong M_n(eKG e).$$

Moreover, $eKG e = e \sum_{j=1}^n s_j K T e = e \sum_{j=1}^n e^{s_j^{-1}} s_j^{-1} K T = e K T$, since e is orthogonal to any different conjugate of itself. So (2.3) gives an isomorphism $\delta : eKG \cong M_n(eKT)$, which satisfies $\delta \left(\sum_{j=1}^n e^{s_j^{-1}} y_j \right) = \text{diag}(e y_1^{s_1}, \dots, e y_n^{s_n})$ for $y_1, \dots, y_k \in K T$.

Set $\hat{\Delta} = \bar{\Delta} \circ \delta$. Then we have

$$\hat{\Delta}(e x) = \bar{\Delta}(\text{diag}(e x^{s_1}, \dots, e x^{s_n})) = \text{diag}(\pi(x^{s_j t_l}) : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq k).$$

Observe that the index i only affects the diagonal entries of $\hat{\Delta}(e x)$ by repeating each entry m times. Furthermore $\{s_j t_l : 1 \leq j \leq n, 1 \leq l \leq k\}$ is a transversal of $G/(Q \times D_p)$. Note that if $x^g \neq x$, then $x^g = ax$ with $1 \neq a \in A$ and so $\pi(x^g) = \pi(a)\pi(x) \neq \pi(x)$ by the choice of π . So, the diagonal of $\hat{\Delta}(e x)$ is formed by $[G : E]$ different entries each with multiplicity $m[E : Q \times D_p]$.

On the other hand we have

$$\begin{aligned} \hat{\Delta}(v) &= \hat{\Delta} \left(\sum_{i=1}^m \sum_{j=1}^n e_i^{s_j^{-1}} x^{g_{ij}} \right) = \bar{\Delta} \left(\text{diag} \left(\sum_{i=1}^m e_i x^{g_{i1} s_1}, \dots, \sum_{i=1}^m e_i x^{g_{in} s_n} \right) \right) \\ &= \text{diag}(\pi(x^{g_{ij} s_j t_l}) \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq l \leq k) \end{aligned}$$

Let $\{v_1, \dots, v_r\}$ be a transversal of $T/T \cap E$ and set $Y_{j_2} = \{(i, j, l) : g_{ij} s_j t_l \equiv h_{j_2} \pmod{E}\}$ and $\bar{Y}_{j_2} = \{(i, j, l) : g_{ij} s_j v_l \equiv h_{j_2} \pmod{E}\}$. Arguing as in the previous paragraph we deduce that the multiplicity of $\pi(x^{h_{j_2}})$ in $\hat{\Delta}(v)$ is $|Y_{j_2}| = [T \cap E : Q \times D_p] |\bar{Y}_{j_2}|$. As v and $e x$ are conjugate in eKG we deduce that $m[E : Q \times D_p] = [T \cap E : Q \times D_p] |\bar{Y}_{j_2}|$ and therefore $|Y_{j_2}| = m[E : Q \times D_p] / [T \cap E : Q \times D_p] = m[E :$

$T \cap E] = mn_1$. Then $(i, j, l) \mapsto (i, j)$ defines a bijective map $\bar{Y}_{j_2} \rightarrow X_{j_2}$. Therefore $|X_{j_2}| = mn_1$ as desired. \square

Corollary 2.6. *Let G be a finite group and A a normal cyclic subgroup of G containing G' . Let $D = Z(C_G(A))$ and u a torsion unit in $\mathbb{Z}G$ satisfying $\omega_D(u) = 1$. Then u is conjugate in $\mathbb{Q}G$ to an element of D .*

Proof. Since G/A is abelian there exists some $b \in G$ such that $\omega_A(u) = bA$. We claim that $\{n \in D : \varepsilon_n(u) \neq 0\}$ is contained in bA . Indeed, let p be some prime dividing $|u|$. By Theorem 2.5 u_p is conjugate in $\mathbb{Z}_p D$ to some $n_{p,0} \in D$. By statement (1b) of Proposition 1.3, if $\varepsilon_n(u) \neq 0$ then n_p is conjugate to $n_{p,0}$, so $n_p A = n_{p,0} A$. On the other hand we have that $b_p A = \omega_A(u_p)$ is conjugate to $n_{p,0} A = \omega_A(n_{p,0})$, so $b_p A = n_{p,0} A$ and hence $b_p A = n_p A$ for all primes p dividing $|u|$ and all $n \in N$ with $\varepsilon_n(u) \neq 0$. As $|n|$ divides $|u|$, by statement (2a), we deduce that $nA = bA$ and the claim is proved.

Assume that the statement of the corollary is false. Then, by Proposition 1.1 and Lemma 2.3, there exists an $n \in G$ such that $\varepsilon_n(u) < 0$. So $n \in bA$, by the previous paragraph. As in the proof of Theorem 2.5 there exists a subgroup H in D such that $H \cap A = 1$ and D/H is cyclic. So D has a linear character η with kernel H . Then, by Lemma 2.4, we have $0 \leq \sum_{h \in H} [C_G(hn) : D] \varepsilon_{hn}(u)$. But if $\varepsilon_{hn}(u) \neq 0$ with $h \in H_{p'}$ then $hn \in bA \cap nH = nA \cap nH = n(A \cap H) = \{n\}$. Hence

$$0 \leq \sum_{h \in H} [C_G(hn) : D] \varepsilon_{hn}(u) = [C_G(n) : D] \varepsilon_n(u) < 0,$$

a contradiction. \square

3. REDUCTION TO TORSION UNITS OF D -AUGMENTATION 1

In this section we pursue another idea from [Her08a]. In order to reduce the proof of (ZC1) to the case of units of A -augmentation 1, with A a cyclic normal subgroup of G , Hertweck studied the multiplicities of the image of a representation of G induced from a faithful linear character of A . In our study we need to replace A by $D = Z(C_G(A))$. Now D may not be cyclic and in that case it does not have any faithful linear character. Alternatively we consider linear characters of D , whose kernel does not contain any non-trivial normal subgroup of G . Observe that if D is cyclic then the characters satisfying this condition are precisely the faithful linear characters of D .

Let N be an abelian normal subgroup of G . Then for every linear character ψ of N and every $u \in \mathbb{C}G$ one has

$$(3.4) \quad \psi^G(u) = \sum_{n \in N} \psi(n) [C_G(n) : N] \varepsilon_n^G(u),$$

where ψ^G represents the character induced from ψ . This can be checked directly by observing that both sides of the equality define linear maps on $\mathbb{C}G$ and checking the formula for the elements of G . It can be also proved using [Seh93, Lemma 41.10]. If $N = \{n_1, \dots, n_m\}$ and ψ_1, \dots, ψ_m are the linear characters of N then (3.4) yields the following

$$\begin{pmatrix} \psi_1^G(u) \\ \vdots \\ \psi_m^G(u) \end{pmatrix} = T \begin{pmatrix} [C_G(n_1) : N] \varepsilon_{n_1}^G(u) \\ \vdots \\ [C_G(n_m) : N] \varepsilon_{n_m}^G(u) \end{pmatrix},$$

where $T = (\psi_i(n_j))$, the character table of N . By the Orthogonality Relations the transpose conjugate of T is mT^{-1} . Multiplying by this matrix we obtain

$$(3.5) \quad |C_G(x)|\varepsilon_x^G(u) = \sum_{i=1}^m \overline{\psi_i(x)}\psi_i^G(u) \quad \text{for every } x \in N \text{ and } u \in \mathbb{C}G.$$

Let

$$\mathbb{K} = \mathbb{K}_N = \{K \leq N : N/K \text{ is cyclic and } K \text{ does not contain any non-trivial normal subgroup of } G\}.$$

For every $K \in \mathbb{K}$ we select a linear character ψ_K of K with kernel K and let ρ_K be a representation of G affording the induced character ψ_K^G . Observe that if K_1 and K_2 are conjugate in G then $\psi_{K_1}^G = \psi_{K_2}^G$ and therefore we may assume that $\rho_{K_1} = \rho_{K_2}$. Let $\mathcal{C}_{\mathbb{K}}$ be the set of conjugacy classes in G of elements of \mathbb{K} . For every $C \in \mathcal{C}_{\mathbb{K}}$ select a representative K_C of C . Let $\mathbb{Q}_K = \mathbb{Q}(\zeta_{[N:K]})$.

For a square matrix U with entries in \mathbb{C} and $\alpha \in \mathbb{C}$ let $\mu_U(\alpha)$ denote the multiplicity of α as eigenvalue of U . If $U^m = I$ and α is a root of unity then we have the following formula (see [LP89])

$$(3.6) \quad \mu_U(\alpha) = \frac{1}{m} \sum_{d|m} \text{tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}}(\text{tr}(U^d)\alpha^{-d}).$$

This formula is the bulk of the Luthar-Passi Method.

Lemma 3.1. *Let G be a finite group such that (ZC1) holds for every proper quotient of G . Let N be an abelian normal subgroup of G . Let u be a unit of $\mathbb{Z}G$ with $\omega_N(u) \neq 1$ and let $x \in N$.*

(1) *Then*

$$(3.7) \quad |C_G(x)|\varepsilon_x(u) = \sum_{K \in \mathbb{K}} \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}(\overline{\psi_K(x)}\psi_K^G(u)).$$

(2) *Assume moreover that $m = |u|$, $f = |\omega_N(u)|$, $x^m = 1$ and u^d is conjugate in $\mathbb{Q}G$ to an element of G for every $1 \neq d|m$. Then for every $h \mid f$ with $h \neq 1$ we have*

$$(3.8) \quad \sum_{K \in \mathbb{K}} [\mathbb{Q}_K : \mathbb{Q}] \mu_{\rho_K(u)}(\psi_K(x)) = \frac{\varphi(m)}{m} |C_G(x)|\varepsilon_x(u) + \frac{1}{h} \sum_{K \in \mathbb{K}} [\mathbb{Q}_K : \mathbb{Q}] \mu_{\rho_K(u^h)}(\psi_K(x^h))$$

(3) *Assume moreover that G' is cyclic and u^f is conjugate in $\mathbb{Q}G$ to an element y of N . Let $u_C = |\{g \in G : x^f \in y^G K_C^g\}|$ for $C \in \mathcal{C}_{\mathbb{K}}$. Then we have*

$$(3.9) \quad \sum_{K \in \mathcal{C}} \mu_{\rho_K(u^f)}(\psi_K(x^f)) = \frac{[C_G(y) : N]}{|N_G(K_C)|} u_C.$$

Proof. (1) Let ψ be a linear character of N such that the kernel of ψ contains a non-trivial normal subgroup U of G . Then $\psi = \phi \circ \omega_U$ for a linear character ϕ of G/U . By the induction hypothesis $\omega_U(u)$ is conjugate in $\mathbb{Q}(G/U)$ to an element of G/U . Moreover $\omega_{N/U}(\omega_U(u)) = \omega_N(u) \neq 1$ and therefore $\varepsilon_{nU}(\omega_U(u)) = 0$ for every $n \in N$. Then (3.4) yields $\psi^G(u) = \phi^G(\omega_U(u)) = 0$. Hence we can drop in (3.5) all the summands labeled by linear characters of N whose kernel is not in \mathbb{K} . The

remaining characters are those of the form $\sigma \circ \psi_K$ for a $K \in \mathbb{K}$ and $\sigma \in \text{Gal}(\mathbb{Q}_K/\mathbb{Q})$. Hence

$$|C_G(x)|\varepsilon_x(u) = \sum_{K \in \mathbb{K}} \sum_{\sigma \in \text{Gal}(\mathbb{Q}_K/\mathbb{Q})} \overline{\sigma \circ \psi_K(x)} (\sigma \circ \psi_K^G)(u) = \sum_{K \in \mathbb{K}} \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}(\overline{\psi_K(x)} \psi_K^G(u)),$$

as desired.

(2) Let d be a divisor of m such that $d \neq 1$ and $\omega_N(u^d) \neq 1$. By hypothesis u^d is conjugate to an element of G which does not belong to N , by Remark 1.2. Therefore $\varepsilon_n(u^d) = 0$ for every $n \in N$. Hence, by (3.7) we have $\sum_{K \in \mathbb{K}} \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}((\psi_K^G)(u^d) \psi_K(n)^{-1}) = 0$. Thus,

$$(3.10) \quad \text{if } f \nmid d|m, d \neq 1 \text{ and } n \in N \text{ then } \sum_{K \in \mathbb{K}} \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}((\psi_K^G)(u^d) \psi_K(n)^{-1}) = 0.$$

For every $K \in \mathbb{K}$ and every integer d we use the notation $\mu(K, d) = \mu_{\rho_K(u^d)}(\psi_K(x^d))$, the multiplicity of $\psi_K(x^d)$ as an eigenvalue of $\rho_K(u^d)$. By (3.6), for every $e|m$ we have

$$(3.11) \quad \sum_{K \in \mathbb{K}} [\mathbb{Q}_K : \mathbb{Q}] \mu(K, e) = \frac{e}{m} \sum_{K \in \mathbb{K}} \sum_{d|(m/e)} [\mathbb{Q}_K : \mathbb{Q}] \text{tr}_{\mathbb{Q}(\zeta_m^{ed})/\mathbb{Q}}(\psi_K^G(u^{ed}) \psi_K(x)^{-ed}).$$

Let $d|m$ and $\alpha = \psi_K^G(u^d) \psi_K(x)^{-d}$. Clearly the image of ψ_K is \mathbb{Q}_K and $\psi_K^G(u^d) \in \mathbb{Q}(\zeta_m^d)$. Moreover, $x^m = 1$, by hypothesis. Thus $\alpha \in \mathbb{Q}_K \cap \mathbb{Q}(\zeta_m^d)$. Let $L = \mathbb{Q}_K(\zeta_m^d)$. Then $[L : \mathbb{Q}_K] \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}(\alpha) = (\text{tr}_{\mathbb{Q}_K/\mathbb{Q}} \circ \text{tr}_{L/\mathbb{Q}_K})(\alpha) = \text{tr}_{L/\mathbb{Q}}(\alpha) = (\text{tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}} \circ \text{tr}_{L/\mathbb{Q}(\zeta_m^d)})(\alpha) = [L : \mathbb{Q}(\zeta_m^d)] \text{tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}}(\alpha)$. Therefore $[\mathbb{Q}(\zeta_m^d) : \mathbb{Q}] \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}(\alpha) = [\mathbb{Q}_K : \mathbb{Q}] \text{tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}}(\alpha)$. This equality together with (3.7) yields

$$\begin{aligned} \sum_{K \in \mathbb{K}} [\mathbb{Q}_K : \mathbb{Q}] \text{tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}}(\psi_K^G(u^d) \psi_K(x)^{-d}) &= \\ [\mathbb{Q}(\zeta_m^d) : \mathbb{Q}] \sum_{K \in \mathbb{K}} \text{tr}_{\mathbb{Q}_K/\mathbb{Q}}(\psi_K^G(u^d) \psi_K(x)^{-d}) &= \\ [\mathbb{Q}(\zeta_m^d) : \mathbb{Q}] |C_G(x^d)| \varepsilon_{x^d}(u^d). \end{aligned}$$

Moreover, by (3.10), this is 0 provided $f \nmid d|m$ and $d \neq 1$. Thus, for $e = 1$ and $1 \neq h|f$, (3.11) can be reduced to the following

$$\begin{aligned} \sum_{K \in \mathbb{K}} [\mathbb{Q}_K : \mathbb{Q}] \mu(K, 1) &= \\ \frac{[\mathbb{Q}(\zeta_m) : \mathbb{Q}] |C_G(x)|}{m} \varepsilon_x(u) + \frac{1}{m} \sum_{K \in \mathbb{K}} \sum_{h|d|m} [\mathbb{Q}_K : \mathbb{Q}] \text{tr}_{\mathbb{Q}(\zeta_m^d)/\mathbb{Q}}(\psi_K^G(u^d) \psi_K(x)^{-d}) &= \\ \frac{\varphi(m) |C_G(x)|}{m} \varepsilon_x(u) + \frac{1}{h} \frac{h}{m} \sum_{K \in \mathbb{K}} \sum_{d|(m/h)} [\mathbb{Q}_K : \mathbb{Q}] \text{tr}_{\mathbb{Q}(\zeta_m^{hd})/\mathbb{Q}}(\psi_K^G(u^{hd}) \psi_K(x)^{-hd}) &= \\ \frac{\varphi(m) |C_G(x)|}{m} \varepsilon_x(u) + \frac{1}{h} \sum_{K \in \mathbb{K}} [\mathbb{Q}_K : \mathbb{Q}] \mu(K, h), \end{aligned}$$

where in the last equality we have used (3.11) for $e = h$. This proves (3.8).

(3) Finally assume that G' is cyclic and u^f is conjugate in $\mathbb{Q}G$ to $y \in N$. Then $\rho_K(u^f)$, $\rho_K(y)$ and $\text{diag}(\psi_K(y^g)) : g \in T$ are conjugate in the matrices over \mathbb{C} ,

where T is a transversal of G/N . Observe that $\psi_K(y^g) = \psi_K(y^h)$ if and only if $\psi_K((y, g)) = \psi_K((y, h))$ if and only if $(y, g)(y, h)^{-1} \in K$ if and only if $(y, g) = (y, h)$ (because $K \cap G' = 1$), if and only if $gh^{-1} \in C_G(y)$. Therefore each eigenvalue of $\rho_K(u^f)$ has multiplicity $[C_G(y) : N]$. On the other hand $\psi_K(x^f)$ is an eigenvalue of $\rho_K(u^f)$ if and only if $\psi_K(x^f) = \psi_K(y^g)$ for some $g \in G$, if and only if $x^f \in y^G K$. Therefore if $C \in \mathcal{C}_{\mathbb{K}}$ and K_C is a representative of C then

$$\sum_{K \in C} \mu_{\rho_K(u^f)}(\psi_K(x^f)) = \frac{1}{|N_G(K_C)|} \sum_{g \in G, x^f \in y^G K_C^g} [C_G(y) : N] = \frac{[C_G(y) : N]}{|N_G(K_C)|} u_C,$$

as desired. \square

Remark 3.2. Let A be a cyclic normal subgroup of G containing G' . Clearly every element of \mathbb{K} does not intersect A and $Z(G)$. Conversely, let H be a subgroup of G containing a non-trivial normal subgroup U of G and such that $H \cap Z(G) = 1$. If $1 \neq n \in U$ then $1 \neq (n, g) \in A \cap U$ for some $g \in G$ and therefore $A \cap H \neq 1$. Thus, for every abelian subgroup N of G we have $\mathbb{K}_N = \{K \leq N : A \cap K = Z(G) \cap K = 1 \text{ and } N/K \text{ is cyclic}\}$.

Observe that \mathbb{K}_N can be empty. For example, this is the case if $N \cap Z(G)$ is not cyclic.

Lemma 3.3. *Assume that A is a cyclic subgroup of G containing G' . Let N be an abelian subgroup of G containing A and $\mathbb{K} = \mathbb{K}_N$. Then for every $K \in \mathbb{K}$ we have $|\mathbb{K}| \leq |K| = \frac{|N|}{\exp(N)}$.*

Proof. Write $N = C \times H$ with C cyclic of maximal order in N and selected in such a way that if p is prime and $\exp(N_p) = \exp(A_p)$, then $C_p = A_p$. We claim that if $K \in \mathbb{K}$, then $C \cap K = 1$. Otherwise $C_p \cap K \neq 1$ for some prime p and therefore $\exp(C_p) = \exp(N_p) > \exp(A_p)$. Let x be a generator of C_p , $q = |A_p|$ and $a = (x, g)$ with $g \in G$. Then $a \in A_p$ and therefore $a^q = 1$. Thus $(x^q)^g = x^q$. This proves that x^q is a non-trivial central element of G . Then $Z(G) \cap K \neq 1$, contradicting the fact that K does not contain any normal subgroup of G . This proves the claim.

Let π_1 and π_2 be the projections $N \rightarrow C$ and $N \rightarrow H$ along the decomposition $N = C \times H$. By the previous paragraph $K \cap \ker \pi_2 = 1$ and therefore $|K| \geq |H| = \frac{|N|}{\exp(N)}$. As N/K is cyclic we have $\exp(N) \leq [N : K] = \exp(N/K) \leq \exp(N)$. Hence $|K| = \frac{|N|}{\exp(N)} = |H|$ and therefore $\pi_2|_K : K \rightarrow H$ is an isomorphism for every $K \in \mathbb{K}$. Therefore $K = \{f(h)h : h \in H\}$ for a homomorphism $f : H \rightarrow C$. (More precisely $f = \pi_1 \circ \pi_2|_K^{-1}$.) Thus K is completely determined by f and hence $|\mathbb{K}| \leq |\text{Hom}(H, C)| = |H|$. The last equality follows easily from the fact that C is cyclic and $\exp(H)$ divides the order of C . \square

We are ready to prove our main result.

Theorem 3.4. *If G is a cyclic-by-abelian finite group then every normalized torsion unit of $\mathbb{Z}G$ is conjugate in $\mathbb{Q}G$ to an element of G .*

Proof. By means of contradiction we assume that G is a counterexample of minimal order of the theorem and u is a normalized torsion unit of minimal order of $\mathbb{Z}G$ which is not conjugate to an element of G in $\mathbb{Q}G$. We select a cyclic subgroup A of G with G/A abelian and take $D = Z(C_G(A))$ and $\mathbb{K} = \mathbb{K}_D$. By Proposition 1.1, we may assume without loss of generality that $\varepsilon_x(u) < 0$ for some $x \in G$. This implies that the order of x divides the order of u by statement (2a) of Proposition 1.3. Set

$m = |u|$ and $f = |\omega_D(u)|$. By assumption u^d is conjugate in $\mathbb{Q}G$ to an element of G for every $1 \neq d \mid m$.

By Lemma 2.3, $x \in D$ and by Corollary 2.6, $\omega_D(u) \neq 1$. Thus $1 \neq f \mid m$ and in particular, u^f is conjugate in $\mathbb{Q}G$ to some $y \in D$. By the first induction hypothesis (ZC1) holds for every proper quotient of G and hence we can use Lemma 3.1 for $N = D$, u and x . Since $|K| = \frac{|D|}{\exp(D)}$ for every $K \in \mathbb{K}$, by Lemma 3.3, and $[\mathbb{Q}K : \mathbb{Q}] = \varphi([D : K])$ we can write (3.8) for $h = f$ as

$$(3.12) \quad \sum_{K \in \mathbb{K}} \mu_{\rho_K(u)}(\psi_K(x)) = \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])} \varepsilon_x(u) + \frac{1}{f} \sum_{K \in \mathbb{K}} \mu_{\rho_K(u^f)}(\psi_K(x^f)).$$

We claim that

$$(3.13) \quad \frac{1}{f} \sum_{K \in \mathbb{K}} \mu_{\rho_K(u^f)}(\psi_K(x^f)) \leq \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}.$$

Write $f = f_1 f_2$ with f_1 and f_2 positive integers such that the prime divisors of f_1 divide $|D|$ and $(f_2, |D|) = 1$. Then $m = f_2 m'$ with all prime divisors of m' dividing $|D|$. Note that $\langle x^f \rangle = \langle x^{f_1} \rangle$ and so $C_G(x^f) = C_G(x^{f_1})$. Consider the map $\alpha : C_G(x^{f_1}) \rightarrow A$ given by $g \mapsto (x, g)$. If $a = (x, g)$ then $x^g = ax$ and therefore $x^{f_1} = (x^{f_1})^g = a^{f_1} x^{f_1}$. Hence the image of α is contained in $\{a \in A : a^{f_1} = 1\}$ and this is a subgroup of A of order $\leq f_1$. On the other hand $\alpha(g) = \alpha(h)$ if and only if $gh^{-1} \in C_G(x)$. Therefore

$$(3.14) \quad [C_G(x^{f_1}) : C_G(x)] \leq f_1.$$

Assume that $K \in \mathbb{K}$ and y_1 and y_2 are elements of G in the same conjugacy class such that $y_1 K = y_2 K$. Then $y_2 \in y_1 A \cap y_1 K = \{y_1\}$ because $A \cap K = 1$. Therefore, if $C \in \mathcal{C}_{\mathbb{K}}$, then $\{g \in G : (x^f)^g \in y^G K_C\}$ is the disjoint union of the subsets $X_{C, y_1} = \{g \in G : (x^f)^g \in y_1 K_C\}$ with $y_1 \in y^G$. If $g, h \in X_{C, y_1}$ then $(x^f)^{gh^{-1}} = ((x^f)^h k)^{h^{-1}} = x^f k^{h^{-1}}$ for some $k \in K_C$. Then $(x^f, gh^{-1}) \in A \cap K^{h^{-1}} = 1$ and hence $gh^{-1} \in C_G(x^f)$. Conversely, if $gh^{-1} \in C_G(x^f)$ and $g \in X_{C, y_1}$ then $h \in X_{C, y_1}$. This proves that if X_{C, y_1} is not empty, then it is a coset of $C_G(x^f) = C_G(x^{f_1})$. Therefore for u_C as in Lemma 3.1 we get

$$(3.15) \quad \begin{aligned} u_C &= \sum_{y_1 \in y^G} |X_{C, y_1}| \leq |C_G(x^{f_1})| |y^G| = |C_G(x^{f_1})| [G : C_G(y)] \\ &\leq f_1 |C_G(x)| [G : C_G(y)], \end{aligned}$$

and hence

$$(3.16) \quad \sum_{C \in \mathcal{C}_{\mathbb{K}}} \frac{u_C}{|N_G(K_C)|} \leq f_1 \frac{|C_G(x)|}{|C_G(y)|} \sum_{C \in \mathcal{C}_{\mathbb{K}}} [G : N_G(K_C)] = f_1 \frac{|C_G(x)|}{|C_G(y)|} |\mathbb{K}|.$$

Thus

$$(3.17) \quad \frac{1}{f} [C_G(y) : D] \sum_{C \in \mathcal{C}_{\mathbb{K}}} \frac{u_C}{|N_G(K_C)|} \leq \frac{1}{f_2} |\mathbb{K}| [C_G(x) : D].$$

By Lemma 3.3, $|\mathbb{K}| \leq |K|$. Moreover, every prime divisor of m' divides $\exp(D) = [D : K]$ and therefore

$$(3.18) \quad \frac{1}{f_2} \leq \frac{\varphi(f_2)}{f_2} \frac{\varphi(m')}{m'} \frac{[D : K]}{\varphi([D : K])} = \frac{\varphi(m)}{m} \frac{[D : K]}{\varphi([D : K])}.$$

Thus

$$(3.19) \quad \frac{1}{f_2} |\mathbb{K}| [C_G(x) : D] \leq \frac{\varphi(m)}{m} \frac{[D : K]}{\varphi([D : K])} |K| [C_G(x) : D] = \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}.$$

Combining (3.9), (3.17) and (3.19) we conclude that

$$(3.20) \quad \begin{aligned} \frac{1}{f} \sum_{k \in \mathbb{K}} \mu_{\rho_K(u^f)}(\psi_K(x^f)) &\leq \frac{1}{f} [C_G(y) : D] \sum_{C \in \mathcal{C}_K} \frac{u_C}{|N_G(K_C)|} \\ &\leq \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}. \end{aligned}$$

This proves (3.13).

As the left side of (3.12) is non-negative we have

$$\frac{1}{f} \sum_{k \in \mathbb{K}} \mu_{\rho_K(u^f)}(\psi_K(x^f)) \geq -\frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])} \varepsilon_x(u) \geq \frac{\varphi(m)}{m} \frac{|C_G(x)|}{\varphi([D : K])}$$

because $\varepsilon_x(u) < 0$. Therefore the equality holds in (3.13) and therefore the equality holds in all the inequalities from (3.14) to (3.20). This has the following consequences: $\varphi(f_2) = 1$, so that $f_2 \leq 2$; $|C_G(x^{f_1})| = f_1 C_G(x)$ and hence the conjugacy class of x contains all the elements of the form $a^i x$, where a is an element of A of order f_1 ; m is divisible by all the primes dividing $|D|$; $X_{C, y_1} \neq \emptyset$ for every $y_1 \in y^G$; the left hand side of (3.12) is zero and hence $\psi_K(x)$ is not an eigenvalue of $\rho_G(u)$ for every $K \in \mathbb{K}$. Applying this to conjugates of x we deduce that $\psi_K(x^g)$ is not an eigenvalue of $\rho_K(u)$ for every $g \in G$ and every $K \in \mathbb{K}$. We claim that $f_2 \geq 0$ and all the eigenvalues of $\rho_K(u)$ have even order. If ξ is an eigenvalue of $\rho_K(u)$ then ξ^f is an eigenvalue of both $\rho_K(u^f)$ and $\rho_K(y)$. Thus $\xi^f = \psi_K(y_1)$ for some $y_1 \in y^G$. As $X_{C, y_1} \neq \emptyset$, $\xi^f = \psi_K((x^f)^g)$ for some $g \in G$ and therefore $\xi = \zeta_f^j \psi_K(x^g)$ for some j . However, $a^i x^g$ is conjugate to x^g for every $0 \leq i < f_1$, where a is an element of A of order f_1 . Hence $\psi_K(a^i x^g) = \zeta_{f_1}^i \psi_K(x^g)$ is not an eigenvalue of $\rho_K(u)$ for every $0 \leq i < f_1$. This implies that $f_2 = 2$ and j is odd. Therefore the order of ξ is even. This proves the claim. Then all the eigenvalues of $\rho_K(u^{\frac{m}{2}})$ are equal to -1 and hence $\psi_K^G(u^{\frac{m}{2}}) = -[G : D]$. However $u^{\frac{m}{2}}$ is conjugate to an element of $G \setminus D$ and therefore $\psi_K^G(u^{\frac{m}{2}}) = 0$, a contradiction. \square

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 MURCIA, SPAIN

E-mail address: mauriciojc02@hotmail.com

FACHBEREICH MATHEMATIK, UNIVERSITÄT STUTTGART, PFAFFENWALDRING 57, 70569 STUTTGART, GERMANY

E-mail address: leo.imsueden@yahoo.com

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE MURCIA, 30100 MURCIA, SPAIN

E-mail address: adelrio@um.es